

Solution of spin-boson systems in one and two-dimensional geometry via the asymptotic iteration method

R. Koç^a, O. Özer^b, H. Tütüncüler^c, and R.G. Yıldırım^d

Department of Engineering Physics, Faculty of Engineering, University of Gaziantep, 27310 Gaziantep, Turkey

Received 9 November 2006 / Received in final form 26 June 2007

Published online 1st November 2007 – © EDP Sciences, Società Italiana di Fisica, Springer-Verlag 2007

Abstract. We consider solutions of the 2×2 matrix Hamiltonian of physical systems within the context of the asymptotic iteration method. Our technique is based on transformation of the associated Hamiltonian in the form of the first order coupled differential equations. We construct a general matrix Hamiltonian which includes a wide class of physical models. The systematic study presented here reproduces a number of earlier results in a natural way as well as leading to new findings. Possible generalizations of the method are also suggested.

PACS. 03.65.Ge Solutions of wave equations: bound states – 03.65.Ca Formalism – 73.21.La Quantum dots – 71.70.Ej Spin-orbit coupling, Zeeman and Stark splitting, Jahn-Teller effect

Introduction

During the last decade, a great deal of attention has been paid to examining different quantum optical models [1–5]. Recently an iteration technique [6,7] has been suggested to solve the Schrödinger equation which improves both analytical and numerical solutions of the problems and has been developed for some quantum optical systems. The solution of quantum optical Hamiltonians are important in the existing literature. In general, the study of two level-systems in one and two-dimensional geometry coupled to bosonic modes has been the subject of intense attention because of its extensive applicability in various fields of physics [8–18]. Due to the practical and technological importance of these models, it is not surprising that various aspects have been studied both analytically and numerically [19–27]. Such systems have often been analyzed using numerical methods, because the implementation of the analytical techniques does not yield simple analytical expressions. Remarkably, exact solutions have not been thus far presented except for special cases, even though it has been suggested that the problem may be solved exactly. The required Analytical treatments need tedious calculations.

Quantum optical models are one of the most fascinating phenomena in modern physics and chemistry, providing a general approach to understanding the properties of

molecules, crystals and their origins. Most of these Hamiltonians have yet to be solved exactly. Therefore, the natural question arising at this point is: can the asymptotic iteration method be applied and used to obtain the solutions to these systems? The answer of this question is the main topic of this paper.

In recent years much attention has been focused on the asymptotic iteration method (AIM) [6,28–32]. This method reproduces exact solutions to many exactly solvable differential equations and these equations can be related to the Schrödinger equation. It also gives accurate results for non-solvable Schrödinger equations, such as the sextic oscillator, cubic oscillator, deformed Coulomb potential, etc. which are important in applications to many problems in physics. Encouraged by its satisfactory performance through comparison with other methods, we feel tempted to extend AIM to solve matrix differential equations. Although AIM has been applied to solve the Schrödinger equation, its application to the solution of the matrix equations [7] still needs to be improved. In contrast to the solution of the Schrödinger equation including potentials of Coulomb, Morse, harmonic oscillator etc. by using AIM, study of the matrix Hamiltonians has not attracted much attention in the literature. Therefore, we concentrate on the solution of a general two-dimensional two mode bosonic Hamiltonian by using AIM in this paper.

The paper is organized as follows. In Section 2 we discuss transformation of the Hamiltonians whose original forms are given as boson and fermion operators. We show that the Hamiltonian can be expressed as two coupled first order differential equations in the Bargmann-Fock space.

^a e-mail: koc@gantep.edu.tr

^b Corresponding author: e-mail: ozer@gantep.edu.tr

^c e-mail: tutunculer@gantep.edu.tr

^d e-mail: ozkan@gantep.edu.tr

In Section 3, we develop AIM to obtain eigenvalues and eigenfunctions of the different matrix Hamiltonians widely used in physics. Section 4 is devoted to solve a wide class of physical Hamiltonians in the framework of the AIM. Finally we conclude our results in Section 5.

Two dimensional two mode bosonic Hamiltonian

The most general form of the Hamiltonian consists of the coupling of a single spin-1/2 to the boson field in two dimensional geometry. It can be written as

$$H = H_0 + \omega_0 \sigma_0 + (\kappa_1 a + \kappa_2 a^+ + \kappa_3 b + \kappa_4 b^+) \sigma_+ + (\gamma_1 a + \gamma_2 a^+ + \gamma_3 b + \gamma_4 b^+) \sigma_- \quad (1)$$

where $H_0 = \hbar\omega_1 (a^+ a) + \hbar\omega_2 (b^+ b)$ and ω_i, κ_i and γ_i are physical constants. The Pauli matrices $\sigma_{0,\pm}$ are given by

$$\sigma_0 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}; \quad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad (2)$$

and bosonic annihilation a, b and creation a^+, b^+ operators satisfy the usual commutation relations

$$[a, a^+] = [b, b^+] = 1; \quad [a, b] = [a^+, b^+] = [a^+, b] = [a, b^+] = 0. \quad (3)$$

It is seen that the Hamiltonian (1) includes various physical Hamiltonians depending on the choice of the parameters, and it is Hermitian if $\kappa_1 = \gamma_2$, $\kappa_2 = \gamma_1$, $\kappa_3 = \gamma_4$ and $\kappa_4 = \gamma_3$ for real parameters. In general H describes spin phonon (photon) relaxation in the presence of a magnetic field. It can also be used to study interaction of a two-level atom with an electromagnetic field. As explicit examples, when $\omega_1 = \omega_2 = \omega$, $\kappa_2 = \kappa_3 = \gamma_1 = \gamma_4 = 0$ and $\kappa_1 = \kappa_4 = \gamma_2 = \gamma_3 = \kappa$, the Hamiltonian H reduces to the $E \otimes \varepsilon$ Jahn-Teller (JT) Hamiltonian [9,33] and when $\omega_1 = \omega + \omega_c$, $\omega_2 = \omega - \omega_c$, $\kappa_2 = \kappa_3 = \gamma_1 = \gamma_4 = 0$ and $\kappa_1 = -\kappa_4 = \gamma_2 = -\gamma_3 = \kappa$ then the Hamiltonian H becomes the Hamiltonian of quantum dots including spin-orbit coupling [16,12]. One can also obtain Jaynes-Cummings (JC) Hamiltonian [14] and modified JC Hamiltonian [34] as well as many other interesting physical Hamiltonians by appropriate choices of the parameters ω_i, κ_i and γ_i in (1). There exists a relatively large number of different approaches for the solution of the eigenvalue problem for $H\psi = E\psi$ in the literature. However, we present here a systematic treatment for the determination of the eigenvalues and eigenfunctions of (1) in the context of AIM.

We note that the Hamiltonian of a physical system is given in the form of a differential equation in some cases. One way to obtain the bosonic form of a Hamiltonian is to construct a suitable differential form of the bosons. Therefore, it is worth discussing some useful differential forms of the bosons and the connection between them before we begin to present a procedure to solve (1).

Forms of bosons

We are interested in the two-level systems in a one and two-dimensional geometry whose Hamiltonians are given in terms of bosons-fermions or matrix-differential equations. By the use of the differential form of the operators one can easily find the interrelation between boson-fermion and matrix differential equation formalisms of the Hamiltonians. There are various differential forms of the boson operators. At this point, let us start by introducing the following differential forms of the boson operators:

$$\begin{aligned} a^+ &= \frac{\ell}{2}(x + iy) - \frac{1}{2\ell}(\partial_x + i\partial_y), \\ a &= \frac{\ell}{2}(x - iy) + \frac{1}{2\ell}(\partial_x - i\partial_y), \\ b^+ &= \frac{\ell}{2}(x - iy) - \frac{1}{2\ell}(\partial_x - i\partial_y), \\ b &= \frac{\ell}{2}(x + iy) + \frac{1}{2\ell}(\partial_x + i\partial_y) \end{aligned} \quad (4)$$

where $\ell = \sqrt{\frac{m\omega}{\hbar}}$ is the length parameter. In principle, if a Hamiltonian is expressed by boson operators, one could rely directly on the known formulae of the action of boson operators on a state with a defined number of particles without solving differential equations. Apart from the mentioned method, the Hamiltonians can often not be solved exactly, then we need to develop alternative methods. It is amazingly interesting that the Hamiltonian (1) can be solved within the framework of the AIM when it is transformed into the form of the first order coupled differential equations under the constraints that will be given in equations (11a–11d). Now, we briefly discuss three well known differential forms of the bosons and the relation between them. In order to obtain different forms of the bosons we present a transformation procedure. For the sake of simplicity we take $\hbar = m = \omega = 1$ and then ℓ becomes unity.

Consider the following exponential operator

$$A = \exp [\beta (a^+ b + b^+ a)]. \quad (5)$$

The operator acts on the bosons as follows:

$$\begin{aligned} AaA^{-1} &= a \cos \beta - b \sin \beta; \quad Aa^+A^{-1} = a^+ \cos \beta - b^+ \sin \beta \\ AbA^{-1} &= b \cos \beta + a \sin \beta; \quad Ab^+A^{-1} = b^+ \cos \beta + a^+ \sin \beta. \end{aligned} \quad (6)$$

If we set $\beta = -\pi/4$ and change the variable $y \rightarrow iy$ in the transformation given in (6), we obtain the following relations

$$\begin{aligned} Aa^+A^{-1} &= \frac{1}{\sqrt{2}}(x - \partial_x); \quad AaA^{-1} = \frac{1}{\sqrt{2}}(x + \partial_x), \\ Ab^+A^{-1} &= \frac{1}{\sqrt{2}}(y - \partial_y); \quad AbA^{-1} = \frac{1}{\sqrt{2}}(y + \partial_y). \end{aligned} \quad (7)$$

The other important differential forms of the bosons can be obtained by transforming the bosons with the following operator:

$$G = \exp \left[\frac{\alpha}{2} (a^2 + a^{+2} + b^2 + b^{+2}) \right]. \quad (8)$$

where α is the rotation angle of bosons. The action of the operator on the bosons is given by

$$\begin{aligned} \Gamma a \Gamma^{-1} &= a \cos \alpha - a^+ \sin \alpha; \quad \Gamma a^+ \Gamma^{-1} = a^+ \cos \alpha + a \sin \alpha. \\ \Gamma b \Gamma^{-1} &= b \cos \alpha - b^+ \sin \alpha; \quad \Gamma b^+ \Gamma^{-1} = b^+ \cos \alpha + b \sin \alpha. \end{aligned} \tag{9}$$

For $\alpha = \pi/4$, the boson operators take the form

$$\begin{aligned} a &\rightarrow \frac{1}{\sqrt{2}}(a - a^+) = \frac{d}{dx}; \quad a^+ \rightarrow \frac{1}{\sqrt{2}}(a + a^+) = x \\ b &\rightarrow \frac{1}{\sqrt{2}}(b - b^+) = \frac{d}{dy}; \quad b^+ \rightarrow \frac{1}{\sqrt{2}}(b + b^+) = y. \end{aligned} \tag{10}$$

The last formulation is known as the Bargmann-Fock space description of bosons [35] and this form plays a key role when constructing the one-variable first order matrix differential equation form of (1). When we insert the form (10) in (1), the Hamiltonian H becomes a first order and two variable matrix differential equation. In order to separate the variables, we look for the conserved quantity of the system. After some treatment, we obtain the following conserved quantities:

$$K_1 = a^+ a - b^+ b - \frac{1}{2} \sigma_0, \quad \text{when } \gamma_2 = \gamma_3 = \kappa_1 = \kappa_4 = 0 \tag{11a}$$

$$N_1 = a^+ a + b^+ b + \frac{1}{2} \sigma_0, \quad \text{when } \gamma_1 = \gamma_3 = \kappa_2 = \kappa_4 = 0 \tag{11b}$$

$$K_2 = a^+ a - b^+ b + \frac{1}{2} \sigma_0, \quad \text{when } \gamma_1 = \gamma_4 = \kappa_2 = \kappa_3 = 0 \tag{11c}$$

$$N_2 = a^+ a + b^+ b - \frac{1}{2} \sigma_0, \quad \text{when } \gamma_2 = \gamma_4 = \kappa_1 = \kappa_3 = 0. \tag{11d}$$

We note here that although we obtain four conserved quantities, K_1 conjugates with K_2 and N_1 conjugates with N_2 . They can be provided by a similarity transformation

$$K = K_2 = UK_1U^{-1} \quad \text{and} \quad N = N_2 = UN_1U^{-1} \tag{12}$$

where $U = \sigma_+ + \sigma_-$. It is well known that if two quantum mechanical operators commute then they have common eigenfunctions. Thus one can write

$$\begin{aligned} K |n_1, n_2\rangle &= \left(k + \frac{1}{2}\right) |n_1, n_2\rangle, \\ N |n_1, n_2\rangle &= \left(k + \frac{1}{2}\right) |n_1, n_2\rangle, \end{aligned} \tag{13}$$

and the eigenvalue problem can easily be solved in the Bargmann-Fock space. Thus, we obtain the following expressions for the eigenfunctions of K and N :

$$\psi(x, y) = x^k \phi(xy) |\uparrow\rangle + x^{k+1} \phi(xy) |\downarrow\rangle, \quad \text{for } K \tag{14a}$$

$$\psi(x, y) = x^{k+1} \phi(y/x) |\uparrow\rangle + x^k \phi(y/x) |\downarrow\rangle \quad \text{for } N \tag{14b}$$

where $|\uparrow\rangle$ stands for the up state and $|\downarrow\rangle$ stands for the down state. In this case one can normalize the wavefunction $\psi(x, y)$ by using the relation

$$\int_{-\infty}^{\infty} e^{-W} \psi(x, y) dx dy$$

where W is the weight function and the integral is taken over all space. The eigenfunction of the Hamiltonian can be obtained from the relation

$$|n_1, n_2\rangle = \Gamma^{-1} A^{-1} \psi(x, y). \tag{15}$$

Meanwhile we note that the conserved quantity N crucially depends on the conservation of the number of particles. The classical motion of the particle takes place in the space of angular momentum on a sphere. However, to obtain a physical meaning of K from another perspective; when the motion of the particle takes place on an ellipsoid, the conserved quantity is given in the form of (13). We have obtained four different conserved quantities for the Hamiltonian (1) depending on the choice of parameters. However, it is fact that K_1 and K_2 can easily be mapped into each other. It implies that Hamiltonians obtained under the constraints (11a) and (11c) correspond to physically similar systems. We also say that N_1 and N_2 can also be mapped into each other and the Hamiltonians obtained under constraints (11b) and (11d) also correspond to physically similar systems. Therefore, it is valid to discuss the solution of the Hamiltonian (1) under the conditions given in (11c) and (11d).

Since equations (11a) and (1) commute, then they have the same eigenfunctions under the constraint given in (11a). Thus, insertion of (14a) into the Hamiltonian (1) and using the form (10), we obtain the following set of one variable coupled differential equations:

$$\begin{aligned} &\left[(\omega_1 + \omega_2) z \frac{d}{dz} + \left(k + \frac{1}{2}\right) \omega_1 + \omega_0 - E \right] \phi_1(z) \\ &+ \left[\left(k + 1 + z \frac{d}{dz}\right) \kappa_1 + \kappa_4 z \right] \phi_2(z) = 0 \end{aligned} \tag{16a}$$

$$\begin{aligned} &\left[(\omega_1 + \omega_2) z \frac{d}{dz} + \left(k + \frac{3}{2}\right) \omega_1 + \omega_2 - \omega_0 - E \right] \phi_2(z) \\ &+ \left[\gamma_3 \frac{d}{dz} + \gamma_2 \right] \phi_1(z) = 0 \end{aligned} \tag{16b}$$

where $z = xy$ and E is the eigenvalue of the Hamiltonian H and $\phi_1(z)$ and $\phi_2(z)$ correspond to up and down eigenfunctions of the Hamiltonian H , respectively. Similarly, when we substitute (14b) into the Hamiltonian (1) with the form (10), we obtain the following set of one variable coupled differential equations:

$$\begin{aligned} &\left[(\omega_2 - \omega_1) z \frac{d}{dz} + (k + 1) \omega_1 + \omega_0 - E \right] \phi_1(z) \\ &+ [\kappa_2 + \kappa_4 z] \phi_2(z) = 0 \end{aligned} \tag{17a}$$

$$\left[(\omega_2 - \omega_1) z \frac{d}{dz} + k\omega_1 - \omega_0 - E \right] \phi_2(z) + \left[\gamma_1(k+1) + (\gamma_3 - \gamma_1 z) \frac{d}{dz} \right] \phi_1(z) = 0. \quad (17b)$$

where $z = y/x$. Our task is now to apply the AIM to solve the corresponding Hamiltonian.

Development of the AIM for matrix Hamiltonians

In this section we systematically present a procedure for the solution of 2×2 first-order matrix differential equations. Consider the following first order matrix differential equation:

$$\phi' = u_0 \phi \quad (18)$$

where $\phi = [\phi_1, \phi_2]^T$, two component column vector u_0 is a 2×2 matrix function. Note that ϕ and u_0 are functions of z and ϕ' is the first derivative with respect to z . Now, in order to obtain a general solution to this equation in the framework of the AIM we use similar arguments to those given in [7]. The differential equation (18) can be written as the two coupled equation

$$\phi_1' = a_0 \phi_1 + b_0 \phi_2; \quad \phi_2' = c_0 \phi_2 + d_0 \phi_1 \quad (19)$$

where a_0, b_0, c_0 and d_0 are elements of the matrix u_0 . It is easy to show that the n^{th} derivative of ϕ_1 and ϕ_2 can be written as

$$\begin{aligned} \phi_1'' &= a_1 \phi_1 + b_1 \phi_2; & \phi_2'' &= c_1 \phi_2 + d_1 \phi_1 \\ \phi_1''' &= a_2 \phi_1 + b_2 \phi_2; & \phi_2''' &= c_2 \phi_2 + d_2 \phi_1 \\ &\dots & & \end{aligned} \quad (20)$$

$$\begin{aligned} \phi_1^{(n)} &= a_{n-1} \phi_1 + b_{n-1} \phi_2; & \phi_2^{(n)} &= c_{n-1} \phi_2 + d_{n-1} \phi_1 \\ \phi_1^{(n+1)} &= a_n \phi_1 + b_n \phi_2; & \phi_2^{(n+1)} &= c_n \phi_2 + d_n \phi_1. \end{aligned}$$

In order to discuss the asymptotic properties of (1), it is necessary to determine the coefficients a_n, b_n, c_n and d_n . After some straightforward calculation, we can obtain the following relations:

$$\begin{aligned} a_n &= a_0 a_{n-1} + a'_{n-1} + d_0 b_{n-1} \\ b_n &= b_0 a_{n-1} + b'_{n-1} + c_0 b_{n-1} \\ c_n &= c_0 c_{n-1} + c'_{n-1} + b_0 d_{n-1} \\ d_n &= d_0 c_{n-1} + d'_{n-1} + a_0 d_{n-1}. \end{aligned} \quad (21)$$

Our task is now to introduce the asymptotic aspect of the method. For this purpose, the n th and $(n+1)$ th derivatives of ϕ_1 and ϕ_2 can be written as

$$\begin{aligned} \phi_1^{(n)} &= a_{n-1} \left(\phi_1 + \frac{b_{n-1}}{a_{n-1}} \phi_2 \right), \\ \phi_2^{(n)} &= c_{n-1} \left(\phi_2 + \frac{d_{n-1}}{c_{n-1}} \phi_1 \right) \\ \phi_1^{(n+1)} &= a_n \left(\phi_1 + \frac{b_n}{a_n} \phi_2 \right), \\ \phi_2^{(n+1)} &= c_n \left(\phi_2 + \frac{d_n}{c_n} \phi_1 \right). \end{aligned} \quad (22)$$

The coefficients d_0 and c_0 include the coupling constants. Therefore, for sufficiently large n we can suggest the following asymptotic constraints:

$$\frac{b_{n-1}}{a_{n-1}} = \frac{b_n}{a_n} = \lambda_1; \quad \frac{d_{n-1}}{c_{n-1}} = \frac{d_n}{c_n} = \lambda_2. \quad (23)$$

In this formalism, the relations given in (23) imply that the wave functions ϕ_1 and ϕ_2 are truncated for sufficiently large n and the roots of the relations given in (23) belong to the spectrum of the matrix Hamiltonian. Therefore, one can easily compute the energy of the Hamiltonian by solving (23) for the energy term when $z \rightarrow z_0$. Under the asymptotic condition of (23), one can find the wave functions ϕ_1 and ϕ_2 . When we take $\frac{\phi_1^{(n+1)}}{\phi_1^{(n)}}$ and $\frac{\phi_2^{(n+1)}}{\phi_2^{(n)}}$ by using (22) under the constraints given in (23), we obtain:

$$\phi_1^{(n)} = \exp \left(\int \frac{a_n}{a_{n-1}} dz \right) \quad \text{or} \quad \phi_2^{(n)} = \exp \left(\int \frac{c_n}{c_{n-1}} dz \right). \quad (24)$$

Substituting the expression of a_n (and c_n) given in (21) into (24) and then replacing the $\phi_1^{(n)}$ (and $\phi_2^{(n)}$) in (20), respectively, one gets the following expressions;

$$\begin{aligned} \phi_1 + \lambda_1 \phi_2 &= \exp \left(\int (a_0 + \lambda_1 d_0) dz \right) \\ \text{or} \\ \phi_2 + \lambda_2 \phi_1 &= \exp \left(\int (c_0 + \lambda_2 b_0) dz \right). \end{aligned} \quad (25)$$

Using the second equality in (25), one can substitute ϕ_2 into the first equality in equation (19). Thus, one writes

$$\phi_1' + (b_0 \lambda_2 - a_0) \phi_1 = C_1 b_0 \exp \left(\int (c_0 + \lambda_2 b_0) dz \right) \quad (26)$$

and the solution is found as

$$\begin{aligned} \phi_1 &= \exp \left(\int (a_0 - \lambda_2 b_0) dz \right) \\ &\times \left[\int C_1 b_0 e^{\int (c_0 + \lambda_2 b_0) dz} dz + C_2 \right]. \end{aligned} \quad (27)$$

If the same procedure is performed for ϕ_2 , one finds the solution as

$$\begin{aligned} \phi_2 &= \exp \left(\int (c_0 - \lambda_1 d_0) dz \right) \\ &\times \left[\int C_3 d_0 e^{\int (a_0 + \lambda_1 d_0) dz} dz + C_4 \right]. \end{aligned} \quad (28)$$

An immediate practical consequence of these results is that the eigenvalues and eigenfunctions of the various quantum optical Hamiltonians can be determined. In the following sections, it is shown that this asymptotic approach opens the way to the treatment of a large class of matrix Hamiltonians of practical interest.

Results and Discussions

In this part, we apply the results of previous sections to obtain the solutions of the Hamiltonians given by equations (16a,b) and (17a,b). We briefly discuss the corresponding physical system of each Hamiltonian.

Solution of the Hamiltonian H under the constraints: $\gamma_1 = \gamma_4 = \kappa_2 = \kappa_3 = 0$

In this case the Hamiltonian includes two important physical Hamiltonians: $E \otimes \varepsilon$ Jahn-Teller (JT) Hamiltonian [9] and Hamiltonians of quantum dots including spin-orbit coupling [16,33]. When $\omega_1 = \omega_2 = \omega$, $\kappa_2 = \kappa_3 = \gamma_1 = \gamma_4 = 0$ and $\kappa_1 = \kappa_4 = \gamma_2 = \gamma_3 = \kappa$, the Hamiltonian H reduces to the $E \otimes \varepsilon$ JT Hamiltonian as we have mentioned before. It is obvious that the corresponding first order differential equations (16a, b) for JT Hamiltonian can be written in the following form:

$$\phi'_1 = a_0\phi_1 + b_0\phi_2; \quad \phi'_2 = c_0\phi_2 + d_0\phi_1. \quad (29)$$

where the coefficients a_0, b_0, c_0 and d_0 are given by

$$\begin{aligned} a_0 &= \frac{\kappa^2 - 2\omega_0 - 2k + 2E - 2}{4z - \kappa^2} \\ b_0 &= \frac{\kappa(\omega_0 + k + 2z + E)}{\kappa^2 - 4z} \\ c_0 &= \frac{\kappa^2(1 + k + z) + 2z(\omega_0 - k + E - 2)}{z(4z - \kappa^2)} \\ d_0 &= \frac{\kappa(E - \omega_0 - k + 2z + 1)}{z(\kappa^2 - 4z)}. \end{aligned} \quad (30)$$

Using a simple MATHEMATICA program one can compute a_n, b_n, c_n and d_n by using the relations given in (21). On the other hand, for each iteration the expression $\delta_1(z) = b_{n-1}(z)a_n(z) - a_{n-1}(z)b_n(z)$ (and $\delta_2(z) = d_{n-1}(z)c_n(z) - c_{n-1}(z)d_n(z)$) depends on different variables, such as E_n, κ, ω_0 and z . It is also noticed that the iterations should be terminated by imposing the quantization condition $\delta_i(z) = 0, i = 1, 2$ as an approximation to (23) to obtain the eigenenergies. The calculated eigenenergies E_n by means of this condition should, however, be independent of the choice of z . The choice of z is observed to be critical only to the speed of convergence of the eigenenergies, as well as for the stability of the process. In our study it has been observed that the optimal choice for z is $z = 0$ [31]. Therefore, we set $z = 0$ at the end of the iterations. We also note that the first value of the solution set of $\delta_1(z) = 0$ (or $\delta_2(z) = 0$) is not physically acceptable unless the system is exactly solvable.

To fix the iteration number for convergence, the first twenty energy levels have been determined for $n = 8, 9, 10, 11, 12, 14, 16, 18$ iterations for the Hamiltonian above. It has been obtained that $E_{20} = 21.103745$ for $n = 10$, $E_{20} = 21.007171$ for $n = 11$, $E_{20} = 21.007064$ for $n = 12$, $E_{20} = 21.007064$ for $n = 13$, and $E_{20} = 21.007064$ for $n = 14, 16, 18$ iterations, respectively, for $k = 1, w_0 =$

Table 1. The ground-state energies obtained from references [36–39] and from the quantization condition by using coefficients given in equation (30) for different values of κ in the case that $k = 0, w_0 = 0, w = 1$.

κ	Ref. [36]	Ref. [37]	Ref. [38]	Ref. [39]	Present Results
0.25	0.774	0.7766	0.7765	0.7739	0.7738
0.5	0.578	0.5877	0.5870	0.5780	0.5780
0.75	0.400	0.4173	0.4158	0.3998	0.3997
1	0.233	0.2586	0.2560	0.2331	0.2330
2	-0.369	-0.3157	-0.3232	-0.3686	-0.3689
3	-0.919	-0.8466	-0.8575	-0.9177	-0.9189
5	-1.961	-1.8716	-1.8831	-1.9540	-1.9610
7	-2.976	-2.8833	-2.8932	-2.9586	-2.9760
10	-4.485	-4.3937	-4.4019	-4.4594	-4.4850
15	-6.991	-6.9042	-6.9108	-6.9610	-6.9901
20	-9.493	-9.4111	-9.4168	-9.4627	-9.4809
30	-14.496	-14.4202	-14.4249	-14.4651	-14.488

$0, w = 1, \kappa = \frac{1}{10}$. It is obviously seen that there is no change in the 20th energy value for $n \geq 12$. Since it is also the same for the other Hamiltonians given below, $n = 14$ iteration is assumed to be sufficient for the determination of the energy eigenvalues of the related Hamiltonians.

In Table 1 we have compared our results with previous studies. It is seen that the results obtained by the AIM agree with those in references [36–39].

Under the given conditions the Hamiltonian (1) takes the form:

$$H = H_0 + \omega_0\sigma_0 + \kappa [(a + b^+) \sigma_+ + (a^+ + b) \sigma_-]. \quad (31)$$

As a related topic, we mention here that studies of the $E \otimes \varepsilon$ JT problem led Judd [8] to discover a class of exact isolated solutions of the model. To determine the relations between the parameters of the model, one can obtain an analytic form of two eigenvectors of the Hamiltonian corresponding to the specific energy. The complete description of these solutions has been given by Koç et al. [33]. They observed that quasi-exact solutions can be obtained by using $osp(2, 2)$ super algebra.

The physical systems described by the Hamiltonian (31) are summarized as follows. When $\omega_0 = 1/2$ and $k = 0$, the corresponding equation related to the Hamiltonian of the displaced coupled harmonic oscillator whose eigenvalues are obtained as a function of coupling constant, κ , is solved by using AIM and the result is given in Figure 1.

The Hamiltonian corresponds to three octahedral JT systems when $\omega_0 = 1/2$ and k takes integer or half-integer values. These octahedral systems are $\Gamma_8 \otimes (\varepsilon + \tau_2)$ linear

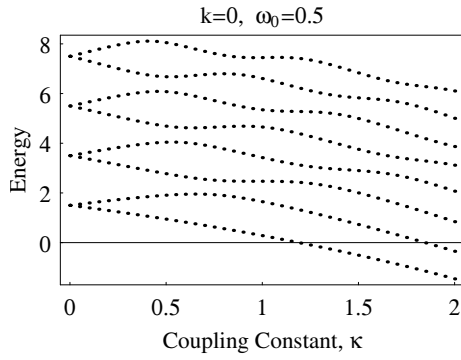


Fig. 1. Energy of the displaced coupled harmonic oscillator as a function of coupling constant.

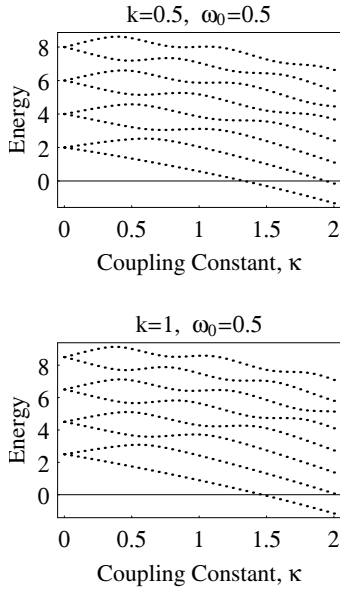


Fig. 2. Energy of the octahedral JT systems as a function of coupling constant.

$E \otimes \varepsilon$ and linear $\Gamma_8 \otimes \tau_2$, for which their eigenvalues are depicted in Figure 2.

Finally, in the presence of an external field, $\omega_0 \neq 1/2$, the Hamiltonian is compatible with the generalized $E \otimes \varepsilon$ JT system and dimers. For dimers $k = 0$ and for generalized $E \otimes \varepsilon$ JT system, k takes half-integer values. The results of the iteration for dimers are given in Figure 3 and for generalized $E \otimes \varepsilon$ JT system are given in Figure 4.

It is shown in the first 4 figures that the energy becomes an oscillating function of the coupling constant κ when $k > 0$.

The Hamiltonians of quantum dots including Rashba coupling can be obtained when we set $\omega_1 = \omega_2 = \omega$, $\kappa_2 = \kappa_3 = \gamma_1 = \gamma_4 = 0$ and $\kappa_1 = -\kappa_4 = \gamma_2 = -\gamma_3 = \kappa$, and the coefficients of the coupled differential equations are

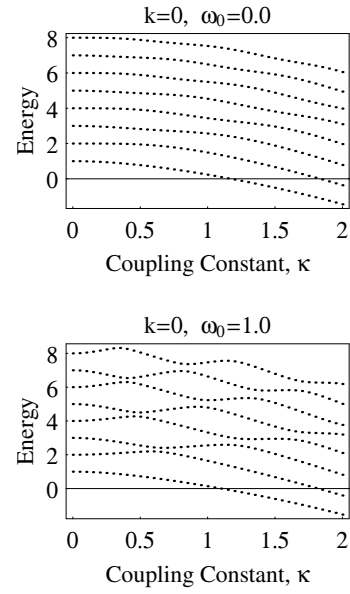


Fig. 3. Energy of the dimer as a function of coupling constant.

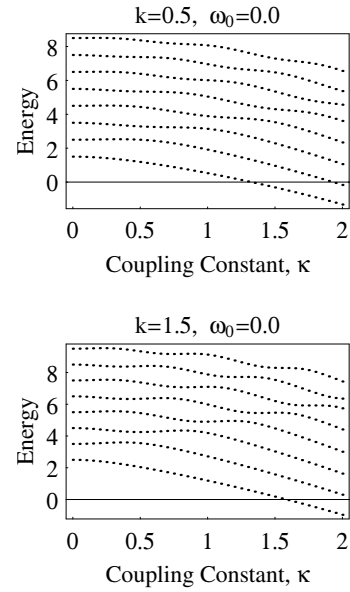


Fig. 4. Energy of the generalized $E \otimes \varepsilon$ JT system as a function of coupling constant.

given by

$$\begin{aligned}
 a_0 &= \frac{\kappa^2 - 2\omega_0 - 2k + 2E - 2}{4z + \kappa^2} \\
 b_0 &= \frac{\kappa(2z - \omega_0 - k - E)}{\kappa^2 + 4z} \\
 c_0 &= \frac{\kappa^2 + 2(\omega_0 + k + E)}{4z + \kappa^2} - \frac{k+1}{z} \\
 d_0 &= \frac{\kappa(E - \omega_0 - k - 2z - 1)}{z(\kappa^2 + 4z)}.
 \end{aligned} \tag{32}$$

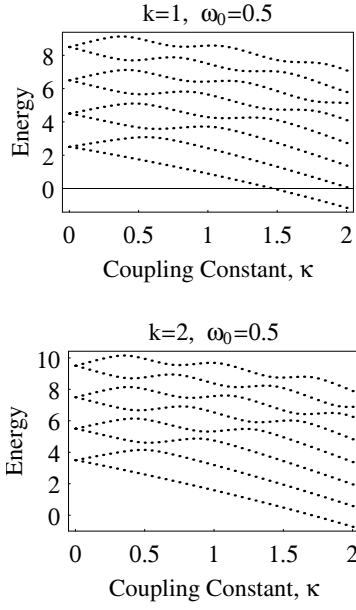


Fig. 5. Energy of the Rashba Hamiltonian as a function of coupling constant.

Note that the origin of the Rashba spin-orbit coupling in quantum dots is due to the lack of inversion symmetry which causes a local electric field perpendicular to the plane of the heterostructure. In the literature, the Hamiltonian has been formalized in the coordinate-momentum space leading to a matrix differential equation.

Solution of the Hamiltonian H under the constraints: $\gamma_1 = \gamma_3 = \kappa_2 = \kappa_4 = 0$

Under the constraints given in this section, we study various well-known exactly solvable Hamiltonians which give us opportunity to test our approach. By the given constraint, the Hamiltonian (1) takes the form

$$H = H_0 + \omega_0 \sigma_0 + (\kappa_1 a + \kappa_3 b) \sigma_+ + (\gamma_2 a^+ + \gamma_4 b^+) \sigma_- \quad (33)$$

We interpret below the solution of the three physical systems using AIM.

Jaynes Cummings Hamiltonian ($\gamma_4 = \kappa_3 = \omega_2 = 0, \kappa_1 = \gamma_2 = \kappa$)

The Jaynes-Cummings (JC) Hamiltonian with rotating wave approximation is given by

$$H = \omega a^+ a + \omega_0 \sigma_0 + \kappa (\sigma_+ a + \sigma_- a^+) \quad (34)$$

In this case the coefficients of the coupled differential equations (17a,b) are given by

$$\begin{aligned} a_0 &= - \left(\frac{\kappa^2 + \omega (E - k\omega + \omega_0)}{\omega^2 z} \right) \\ b_0 &= \frac{\kappa (E - \omega_0)}{\omega^2 z} \\ c_0 &= \frac{-E + \omega + k\omega + \omega_0}{\omega z} \\ d_0 &= \frac{\kappa}{\omega z} \end{aligned} \quad (35)$$

The AIM leads to the following expressions for the eigenvalues of the JC Hamiltonian

$$\begin{aligned} n = 1; \quad E &= \left(k + \frac{1}{2} \right) \omega \mp \frac{1}{2} \sqrt{4\kappa^2 (k+1) + (\omega + 2\omega_0)^2} \\ n = 2; \quad E &= \left(k - \frac{1}{2} \right) \omega \mp \frac{1}{2} \sqrt{4\kappa^2 (k) + (\omega + 2\omega_0)^2} \\ n = 3; \quad E &= \left(k - \frac{3}{2} \right) \omega \mp \frac{1}{2} \sqrt{4\kappa^2 (k-1) + (\omega + 2\omega_0)^2} \\ n = n; \quad E &= \left(k + \frac{3}{2} - n \right) \omega \\ &\quad \mp \frac{1}{2} \sqrt{4\kappa^2 (k+2-n) + (\omega + 2\omega_0)^2} \end{aligned} \quad (36)$$

It is obvious that when the coupling constant κ is zero, then the result is the eigenvalues of the simple harmonic oscillator.

Substituting the variables a_0, b_0, c_0 and d_0 in equations (27) and (28), one finds the eigenfunctions $\phi_1(z)$ and $\phi_2(z)$ as

$$\begin{aligned} n = 1; \quad \phi_1 &= 1 & \phi_2 &= 1 \\ n = 2; \quad \phi_1 &= z & \phi_2 &= z \\ n = 3; \quad \phi_1 &= z^2 & \phi_2 &= z^2 \\ n = n; \quad \phi_1 &= z^{n-1} & \phi_2 &= z^{n-1} \end{aligned} \quad (37)$$

Using equation (14b), one writes

$$\psi(x, y) = x^k (y/x)^{n-1} | \downarrow \rangle + x^{k+1} (y/x)^{n-1} | \uparrow \rangle \quad (38)$$

and finds

$$\psi(x, y) = C x^{1+k-n} y^{n-1} (| \downarrow \rangle + x | \uparrow \rangle) \quad (39)$$

where C is the normalization constant, k is the state number and n is the iteration number. In order to find the original eigenfunction of the Hamiltonian, one can use equation (15) by replacing the operators given in (5) and (8).

Dirac Oscillator ($\omega_1 = \omega_2 = \kappa_3 = \gamma_4 = 0$)

One can also show that the constraints given in this section include the Dirac oscillator. In order to show this we express the Dirac oscillator with boson operators. Consider the $(2+1)$ dimensional Dirac equation for a free particle

of mass m in terms of two component spinors, then ψ can be written as [10,11,40]

$$E\psi = \left(\sum_{i=1}^2 c\sigma_i p_i + mc^2\sigma_0 \right) \psi. \quad (40)$$

The momentum operator p_i is a differential operator $\mathbf{p} = -i\hbar(\partial_x, \partial_y)$ and the 2D Dirac oscillator can be constructed by changing the momentum as $\mathbf{p} \rightarrow \mathbf{p} - im\omega\sigma_0\mathbf{r}$. Then the Dirac equation (40) takes the form

$$(E - mc^2\sigma_0)\psi = c[(p_x - ip_y) - im\omega'(x - iy)]\sigma_+ + c[(p_x + ip_y) - im\omega'(x + iy)]\sigma_-. \quad (41)$$

After some straightforward treatment, we obtain the bosonic form of the Dirac oscillator:

$$(E - mc^2\sigma_0)\psi = 2ic\sqrt{m\omega'\hbar}[a\sigma_+ + a^\dagger\sigma_-]\psi. \quad (42)$$

An immediate practical consequence of these results is the Lie algebraic structure of the Hamiltonians that can easily be determined. It can be obtained by setting the parameters of the JC Hamiltonian to $\omega = 0, \kappa = 2ic\sqrt{m\omega'\hbar}$ and $\omega_0 = mc^2$. Then the eigenvalues of the Dirac oscillator are given by

$$E = \pm \frac{1}{2}\sqrt{4m^2c^4 - 4\hbar\omega'mc^2(k \pm n)}. \quad (43)$$

Modified Jaynes-Cummings Hamiltonian ($\kappa_1 = \gamma_2 = \kappa_3 = \gamma_4 = \kappa, \omega_2 = \omega_1 = \omega$)

In addition to these Hamiltonians, we can also show that our formalism includes another important Hamiltonian: When a single two-level atom is placed in the common domain of two cavities interacting with two quantized modes, the Hamiltonian of such a system can be obtained from the modification of the JC Hamiltonian and is given by

$$H = \omega a^\dagger a + \omega b^\dagger b + \omega_0\sigma_0 + \kappa(a + b)\sigma_+ + \kappa(a^\dagger + b^\dagger)\sigma_-. \quad (44)$$

Without detailed calculations, one can obtain the energy of the MJC Hamiltonian in the closed form by using AIM:

$$E = \left(k + \frac{3}{2} \right) \pm \frac{1}{2}\sqrt{8(k+1-n)\kappa^2 + (2\omega_0 - 1)^2}. \quad (45)$$

Consequently the exact eigenvalues can be reproduced by AIM.

Conclusion

We have systematically discussed the solutions of various physical Hamiltonians within the framework of AIM. We have shown that the formalism given in this paper leads to the exact or approximate solution of the problems of various physical systems. We have applied the AIM to

the problem of an electron in a quantum dot in the presence of both a magnetic field and spin-orbit coupling. The procedure presented here gives an accurate result for the eigenvalues of the both JT and Rashba Hamiltonians. The suggested approach can easily be used to solve other quantum optical problems which are not discussed here.

We have presented a transformation procedure that offers several advantages, especially if one wishes to describe the eigenvalues of the bosonic Hamiltonians by using AIM. We have also presented the steps towards an extension of the AIM.

The technique given in this article can be extended in several ways. The Hamiltonian of a quantum dot including position dependent effective mass may be formulated and solved within the procedure given here. We hope that our method leads to interesting results on the spin-orbit effects in quantum dots in future studies. Along this line we have work in progress.

The authors would like to thank the referees for their remarkable suggestions which improved the presentation of the paper. One of the authors (O. Özer) is also grateful to the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy, for its hospitality.

References

1. G.A. Prinz, *Science* **282**, 1660 (1998)
2. S.A. Wolf, D.D. Awschalom, R.A. Buhrman, J.M. Daughtan, S. vov Molnár, M.L. Roukes, A.Y. Chitchekanova, D.M. Treger, *Science* **294**, 1488 (2001)
3. R. Winkler, *Phys. Rev. B* **62**, 4254 (2000)
4. X.F. Wang, P. Vasilopoulos, F.M. Peeters, *Appl. Phys. Lett.* **80**, 1400 (2002)
5. S. Bandyopadhyay, *Phys. Rev. B* **61**, 13813 (2000)
6. H. Çiftci, R.L. Hall, N. Saad, *J. Phys. A* **36**, 11807 (2003)
7. H. Çiftci, R.L. Hall, N. Saad, *Phys. Rev. A* **72**, 022101 (2005)
8. B.R. Judd, *J. Phys. C* **12**, 1685 (1979)
9. H.G. Reik, M.E. Stülze, M. Dacha, *J. Phys. A* **20**, 6327 (1987)
10. D. Itó, K. Mori, E. Carriere, *Nuovo Cimento A* **51**, 1119 (1967)
11. M. Moshinsky, A. Szczepaniak, *J. Phys. A: Math. Gen.* **22**, L817 (1989)
12. H. Tütüncüler, R. Koç, E. Olğar, *J. Phys. A* **37**, 11431 (2004)
13. E.A. Tur, *Opt. Spectrosc.* **89**, 574 (2000)
14. E.T. Jaynes, F.W. Cummings, *Proc. IEEE* **51**, 89 (1963)
15. G. Dresselhaus, *Phys. Rev.* **100**, 580 (1955)
16. Y.A. Bychkov, E.I. Rashba, *J. Phys. C* **17**, 6039 (1984)
17. E. Tsitsishvili, G. Lozano, A.O. Gogolin, *Phys. Rev. B* **70**, 115316 (2004)
18. D. Frustaglia, K. Richter, *Phys. Rev. B* **69**, 235310 (2004)
19. A.B. Kimov, L.L. Sanchez-Soto, *Phys. Rev. A* **61**, 063802 (2000)
20. V.P. Karassiov, A.A. Gusev, S.I. Vinitzky, *hep-quant/0105152* (2001)

21. V.P. Karassiov, A.B. Klimov, *Phys. Lett. A* **189**, 43 (1994)
22. V. Sunil Kumar, B.A. Bambah, R. Jagannathan, *J. Phys. A* **34**, 8583 (2001)
23. A. Wünsche, *J. Opt. B: Quantum Semiclass. Opt.* **2**, 73 (2000)
24. V. Buzek, *Phys. Rev. A* **29**, 3196 (1989)
25. M. Ban, *J. Math. Phys. B* **33**, 3213 (1992)
26. Y.Q. Chen, *J. Phys. A* **33**, 8071 (2000)
27. A. Balantekin, I. Bars, *J. Phys. A* **23**, 1239 (1982)
28. A.J. Sous, *Chinese J. Phys.* **44**, 167 (2006)
29. H. Çiftci, R.L. Hall, N. Saad, *J. Phys. A* **38**, 1147 (2005)
30. T. Barakat, K. Abodayeh, A. Mukheimer, *J. Phys. A* **38**, 1299 (2005); T. Barakat, *Phys. Lett. A* **344**, 411 (2005)
31. F.M. Fernández, *J. Phys. A* **37**, 6173 (2004)
32. O. Bayrak, I. Boztosun, *J. Phys. A* **39**, 6955 (2006)
33. R. Koç, H. Tütüncüler, M. Koca, E. Körcük, *Prog. Theor. Phys.* **110**, 399 (2003); R. Koç, M. Koca, H. Tütüncüler, *J. Phys. A* **35** 9425 (2002)
34. H. Tütüncüler, R. Koç, *Pramana J. Phys.* **62**, 993 (2004); H. Tütüncüler, R. Koç, *J. Turk. Phys.* **28**, 145 (2004); R. Koç, H. Tütüncüler, M. Koca, E. Olğar, *Ann. Phys.* **319** 333 (2005)
35. V. Bargmann, *Rev. Mod. Phys.* **34**, 829 (1962)
36. H.C. Longuet-Higgins, U. Öpik, M.H.L. Pryce, R.A. Sack, *Proc. Soc. London Ser. A* 244 (1958) 1.
37. C.F. Lo, *Phys. Rev. A* **43**, 5127 (1991)
38. D. Kulak, *Czech. J. Phys.* **52**, 71 (2002)
39. D. Kulak, *Solid State Commun.* **132**, 607 (2004)
40. R. Koç, M. Koca, *Mod. Phys. Lett. A* **20**, 911 (2005); R. Koç, M. Koca, *Turk J. Phys.* **29**, 201 (2005)